

$TMF_0(3)$  CHARACTERISTIC CLASSES FOR STRING BUNDLES

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ABSTRACT. We compute the completed  $TMF_0(3)$  cohomology of the 7-connective cover  $BString$  of  $BO$ . We use cubical structures on line bundles over elliptic curves to construct an explicit class which together with the Pontryagin classes freely generates the cohomology ring.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Characteristic numbers play an important role in the determination of the structure of cobordism rings. For unoriented, oriented and  $Spin$  manifolds the cobordism rings were calculated in the 50s and 60s with the help of Stiefel-Whitney,  $H\mathbb{Z}$ - and  $KO$ -Pontryagin classes (compare [Tho54][Nov62][ABP67]). However, it is known that for manifolds with lifts of the tangential structure to the 7-connective cover  $String$  of  $BO$  these numbers do not determine the bordism classes.

Locally at the prime 2, the Thom spectrum  $MSpin$  splits into summands of connective covers of  $KO$  and an Eilenberg-MacLane part. A similar splitting is conjectured for  $MString$  where  $KO$  is replaced by suitable versions of the spectrum  $TMF$ : the Witten orientation provides a surjection of the  $String$  bordism ring to the ring of topological modular forms and there is evidence that another summand of  $MString$  is provided by the 16 connective cover of  $TMF_0(3)$ . In order to provide maps to this possible summand one has to study  $TMF_0(3)$ -characteristic classes for  $String$  manifolds. This is the subject of this work.

In [Lau] the  $TMF_1(3)$  cohomology rings of  $BSpin$  and  $BString$  were computed. It turned out that the  $Spin$  cohomology ring is freely generated by the Pontryagin classes (see [Lau] for their definition). In the  $String$  case there is another class  $r$  coming up which together with the Pontryagin classes freely generates the cohomology ring when localized at  $K(2)$  for the prime 2.

The theory  $TMF_1(3)$  is a complex orientable theory. Its formal group is the completion of the universal elliptic curve with  $\Gamma_1(3)$  structures. Its relation to  $TMF_0(3)$  is analogous to the relation between complex and Real  $K$ -theory: a  $\Gamma_1(3)$ -structure is a choice of point of exact order 3 on an elliptic curve. A  $\Gamma_0(3)$ -structure is the choice of subgroup scheme of the form  $\mathbb{Z}/3$  of the points of order 3. Given such a subgroup scheme there are exactly two choices of points of exact order 3 and they differ by a sign. Hence the corresponding cohomology theory  $TMF_0(3)$  is the ‘Real’ version of the complex theory  $TMF_1(3)$ . It can be obtained by taking homotopy fixed points under the action which changes the sign of the 3 division point.

It is useful to consider  $TMF_1(3)$  as a Real theory in the sense of Atiyah (compare [Ati66][HK01]), which means that there is a  $\mathbb{Z}/2$ -equivariant spectrum (“the Real

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Date: March 31, 2014.

2000 *Mathematics Subject Classification*. Primary 55N34; Secondary 55P20, 22E66.

theory”) whose non-equivariant restriction (“the complex theory”) is  $TMF_1(3)$  and whose fixed point spectrum (“the Real theory”) is  $TMF_0(3)$ . This allows us to lift the Pontryagin classes to  $TMF_0(3)$  for *Spin* bundles. Our first result is:

**Theorem 1.1.** *There are classes  $\pi_i \in TMF_0(3)^{-32i} BSpin$  which lift the products  $v_2^{6i} p_i$  for the  $TMF_1(3)$  Pontryagin classes  $p_i$ . Moreover, we have*

$$TMF_0(3)^* BSpin \cong TMF_0(3)^* [\pi_1, \pi_2, \dots]$$

The generator  $r$  in the calculation  $TMF_1(3)$  cohomology of  $BString$  has the property that it maps to a generator of  $K(\mathbb{Z}, 3)$  under the canonical map. In fact, it has been shown in [Lau] that any such class can serve as a generator. However, in order to obtain a class  $r$  which is already defined in the Real theory  $TMF_0(3)$  one has to provide a more geometric construction. We use the theory of cubical structures on elliptic curves which also played a role in [AHS01] in the construction of the Witten orientation. We show that a convenient choice of a generator  $r$  is the defect class which compares the Witten orientation with the complex orientation. It turns out that this class admits a lift to the Real theory. Our main result is:

**Theorem 1.2.** *For String bundles  $\xi$  over  $X$  there is a natural stable class*

$$r(\xi) \in TMF_0(3)^0 X$$

*with the following properties:*

- (i)  *$r$  is multiplicative:  $r(\xi \oplus \eta) = r(\xi) \otimes r(\eta)$ .*
- (ii) *There is an isomorphism*

$$\widehat{TMF_0(3)}^* [r, \pi_1, \pi_2, \dots] \longrightarrow \widehat{TMF_0(3)}^* BString$$

*where  $\widehat{TMF_0(3)}$  denotes  $(L_{K(2)} TMF_1(3))^{h\mathbb{Z}/2}$  and, in abuse of notation, the class  $r$  is the  $K(2)$ -local version of the class  $r$  corresponding to the universal bundle over  $BString$ .*

- (iii) *In terms of the Chern character of its elliptic character ( $c$ . [Mil89]) at the cusp  $\infty$  it is given by the formula*

$$ch(\lambda(r(\xi))) = \prod_i \frac{\Phi(\tau, x_i - \omega)}{\Phi(\tau, -\omega)}$$

*where the  $x_i$  are the formal Chern roots of  $\xi \otimes \mathbb{C}$ ,  $\omega = 2\pi i/3$  and  $\Phi$  is the theta function*

$$\begin{aligned} \Phi(\tau, x) &= (e^{x/2} - e^{-x/2}) \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})}{(1 - q^n)^2} \\ &= x \exp\left(-\sum_{k=1}^{\infty} \frac{2}{(2k)!} G_{2k}(\tau) x^{2k}\right). \end{aligned}$$

The paper is organized as follows: we first remind the reader of the theory  $TMF_1(3)$  and construct its Real version in the category of  $\mathbb{Z}/2$ -equivariant spectra. Its homotopy fixed points provides a model for  $TMF_0(3)$ . This allows us to construct Pontryagin classes in the Real world for *String* manifolds. The first result then follows from the homotopy fixed point spectral sequence. Next we use cubical structures to show that the defect class provides a generator in the complex theory. Again the equivariant setting allows us to lift this class to the Real world. Since

the realification of the defect class  $r$  provides a permanent cycle in the homotopy fixed point spectral sequence the main theorem follows.

*Acknowledgements.* The authors like to thank Nitu Kitchloo, Björn Schuster and Vesna Stojanoska for helpful discussions.

## 2. REAL TOPOLOGICAL MODULAR FORMS AND PONTRYAGIN CLASSES

In this section we will construct Pontryagin classes for the Real spectrum of topological modular forms of level 3 and prove the first theorem. We start reviewing level structures for elliptic curves and proceed with the associated spectra of topological modular forms. The method of Hu and Kriz (compare [HK01]) allows a construction of  $TMF_1(3)$  in the category of  $\mathbb{Z}/2$ -equivariant spectra.

Let  $\mathcal{M}$  be the stack of smooth elliptic curves. A morphism  $f : S \rightarrow \mathcal{M}$  determines an elliptic curve  $C_f$  over the base scheme  $S$  (see Deligne and Rapoport [DR75].) There is the line bundle  $\omega$  of invariant differentials on  $\mathcal{M}$ . A modular form of weight  $k$  is section of  $\omega^{\otimes k}$ .

For an elliptic curve  $C$  over a ring  $R$  let  $C[n]$  denote the kernel of the self map  $[n]$  which multiplies by  $n$  on  $C$ . If  $n$  is invertible in  $R$  then étale-locally  $C[n]$  is of the form  $\mathbb{Z}/n \times \mathbb{Z}/n$ . A choice of a subgroup scheme  $A$  of  $C[n]$  which is isomorphic to  $\mathbb{Z}/n$  is called a  $\Gamma_0(n)$  structure and a choice of a monomorphism from  $\mathbb{Z}/n$  to  $C[n]$  is a  $\Gamma_1(n)$  structure.

Moduli problems for  $\Gamma_1(n)$  structures with  $n \geq 3$  are representable whereas for  $\Gamma_0(n)$  structures they are not (compare [KM85]). The case  $n = 3$  can be made more explicit: locally, a curve  $C$  can uniquely be written in the form

$$(2.1) \quad C : y^2 + a_1xy + a_3y = x^3$$

in a way that the distinguished point of order 3  $P$  is the origin  $(0, 0)$  and the invariant differential has the standard form  $\omega = dx/2y + a_1x + a_3$ . This means that the ring of modular forms of level 3 has the form

$$(2.2) \quad M_{\Gamma_1(3)*} \cong \mathbb{Z}[1/3, a_1, a_3, \Delta^{-1}].$$

(see [MR09] for more details).

For étale maps  $f : \text{Spec}(R) \rightarrow \mathcal{M}$  with elliptic curve  $C_f$  there is a spectrum

$$E = \Gamma f^* \mathcal{O}_{TMF}.$$

It is a complex orientable ring spectrum whose formal group  $E^0 BS^1$  is equipped with an isomorphism to the formal completion of  $C_f$  (see [Goe10]). The spectrum  $TMF_1(3)$  can be obtained this way. Its coefficient ring is  $M_{\Gamma_1(3)*}$  and it carries the universal curve  $C$  together with the universal point  $P$  of order 3.

Let  $MU$  be the complex bordism spectrum and let

$$MU_* \cong \mathbb{Z}[x_1, x_2, \dots]$$

for a choice of generators. Let

$$\varphi : MU_* \rightarrow M_{\Gamma_1(3)*}$$

be the map which classifies the formal group with the standard coordinate described above. Define polynomials  $f_i$  in two variables  $x, y$  by

$$\varphi(x_i) = f_i(a_1, a_3).$$

We may assume  $f_1 = x$ ,  $f_3 = y$  and  $f_i \in \mathbb{Z}[x, y]$  since the formal group law of the elliptic curve is defined over  $\mathbb{Z}[a_1, a_3]$ . The sequence  $r_i = x_i - f_i(x_1, x_3)$  is regular in  $MU_*$  and the map

$$MU/r_1, r_2, \dots \longrightarrow TMF_1(3)$$

is an equivalence after inverting  $3\Delta$ .

For  $p = 2$  the Hazewinkel generators for the standard coordinate are  $v_1 = a_1$  and  $v_2 = a_3$  (c.[Lau04].) If one inverts only  $v_2$  and not  $\Delta$ , we have an  $MU$ -module with homotopy  $\mathbb{Z}_{(2)}[v_1, v_2^{\pm 1}]$ . The difference to the (standard) Johnson-Wilson spectrum  $E(2)$  is that the higher Hazewinkel generators may be non-zero. We call such a spectrum a *generalized  $E(2)$*  in analogy with Lawson and Naumann (cf.[LN12]) respectively a *form of  $E(2)$*  in analogy with Mathew [Mat]. Many theorems about the standard  $E(2)$  carry over to generalized  $E(2)$  – often the vanishing of the higher  $v_k$  is not needed in the proofs.

**Convention 2.1.** In the sequel it is not important for us to invert the discriminant  $\Delta$  instead of inverting  $v_2$  only. All of our statements hold for both theories and we won't distinguish them in the notation. Also, all spectra and homotopy groups will be localized respectively completed with respect to  $\mathbb{Z}/2$ .

Next we like to construct the Real version of  $TMF_1(3)$ . Let  $M\mathbb{R}$  be the Real  $MU$  spectrum in the category of  $\mathbb{Z}/2$  equivariant spectra over a complete universe. Hu and Kriz [HK01] show that the canonical map from  $M\mathbb{R}_*$  to  $MU_*$  splits by a map of rings  $\iota$  which sends the generator  $x_i$  of  $MU_*$  to a class of dimension  $i(1 + \alpha)$  for the sign representation  $\alpha$ . Moreover, the action of  $\mathbb{Z}/2$  on  $MU$  which changes the coordinate  $x$  to  $[-1](x)$  corresponds to the  $\mathbb{Z}/2$ -action on  $TMF_1(3)$  with homotopy fixed point set  $TMF_0(3)$ : it comes from the map on the moduli stack which takes the point of order 3 to its negative. As in [HK01], p.332 we may kill the sequence  $\iota(r_1), \iota(r_1), \dots$  in  $M\mathbb{R}_*$  in the category of  $E_\infty M\mathbb{R}$  spectra and invert the periodicity class. We obtain the theory  $\mathbb{E}$  which is a Real version of  $E = TMF_1(3)$ :

**Lemma 2.2.** *Let  $i^*$  be the forgetful functor from  $\mathbb{Z}/2$ -equivariant to non-equivariant spectra. We have*

- (i)  $i^*\mathbb{E} \cong TMF_1(3)$
- (ii)  $\mathbb{E}^{h\mathbb{Z}/2} \cong \mathbb{E}^{\mathbb{Z}/2} \cong TMF_0(3)$

*Proof.* As shown more generally by Hu and Kriz, the spectrum  $\mathbb{E}$  is suitably complete that the homotopy fixed point spectrum  $\mathbb{E}^{h\mathbb{Z}/2}$  coincides with the fixed point spectrum  $\mathbb{E}^{\mathbb{Z}/2}$  up to homotopy, see [HK01], Theorem 4.1 and Comment (4) on p. 349. Compare also with [KW13], Section 8, which has more details. It is easy to check that the proof applies whenever  $v_2$  is invertible (that is, to generalized  $E(2)$  and to the case where  $\Delta$  is inverted). Non-equivariantly, we have  $i^*M\mathbb{R} \cong MU$ , so the other statements follow from what we said before.  $\square$

We can use the canonical  $M\mathbb{R}$ -orientation of  $\mathbb{E}$  to define real Pontryagin classes. However, the construction is less explicit than in the complex case.

**Proposition 2.3.** *The Pontryagin classes  $\pi_i = v_2^{6i} p_i \in TMF_1(3)^{-32i} BSpin$  are real, that is, they lift to  $TMF_0(3)^{-32i} BSpin$ .*

*Proof.* Recall from [KLW04] that the map from  $E^*BU$  over  $E^*BSO$  to  $E^*BSpin$  is surjective for  $E = TMF_1(3)$ . Choose a power series  $q_i$  in the Chern classes which

maps to the Pontryagin class  $p_i$ . Let  $\bar{q}_i$  be a power series with coefficients in  $BP_*$  which maps to  $q_i$  under the map to  $E^*$ .

We write  $B\mathbb{U}$  for the Real space of finite dimensional subspaces of  $\mathbb{C}^\infty$  with the complex conjugation as involution. Then by [HK01] 2.25 and 2.28 we have the canonical isomorphism

$$BP^{2i}BU \cong BP\mathbb{R}^{i(1+\alpha)}B\mathbb{U}.$$

Hence, the class  $\bar{q}_i$  defines a class in  $BP\mathbb{R}^{2i(1+\alpha)}B\mathbb{U}$  and maps to a lift of  $q_i$  in  $\mathbb{E}^{2i(1+\alpha)}B\mathbb{U}$  which we denote again by  $q_i$ . Recall from [KW07] that there is an invertible class  $y \in \mathbb{E}^{-17,-1}$ . As in [KW] consider the product

$$y^{2i}q_i \in \mathbb{E}^{-16j,0}B\mathbb{U}.$$

This class lifts  $v_2^{6i}q_i \in E^{-32i}BU$  and may be mapped to the fixed points

$$\mathbb{E}^{(-32i,0)}B\mathbb{U} \longrightarrow (E^{\mathbb{Z}/2})^{-32i}BO.$$

Its restriction to  $(E^{\mathbb{Z}/2})^{-32i}BSpin$  defines a lift of  $v_2^{6i}p_i$ .  $\square$

The  $\mathbb{Z}/2$ -action on  $TMF_1(3)$  induces an action on  $TMF_1(3)^{BSpin}$  with homotopy fixed point set  $TMF_0(3)^{BSpin}$ . Hence we can use a homotopy fixed point spectral sequence to compute the homotopy groups  $TMF_0(3)^*BSpin$ , and we can compare with Mahowald-Rezk [MR09] who have used the homotopy fixed point spectral sequence to compute  $\pi_*TMF_0(3)$ .

From 2.3 we know that the Pontryagin classes  $\pi_i$  are invariant — a more geometric proof of this property is given in the appendix — and we are well prepared for the proof of the first theorem.

*Proof of 1.1.* The homotopy fixed point spectral sequence has the form

$$E_2^{s,t} = H^s(\mathbb{Z}/2\mathbb{Z}, E^tBSpin) \Rightarrow (E^{h\mathbb{Z}/2\mathbb{Z}})^*BSpin.$$

By [Lau] we have

$$E^*BSpin \cong \mathbb{Z}_2[a_1, a_3, \Delta^{-1}][[\pi_1, \pi_2, \dots]],$$

and  $\mathbb{Z}/2\mathbb{Z}$  acts by  $a_i \mapsto -a_i, \pi_i \mapsto \pi_r$ . Hence the situation is very similar to [MR09] where the spectral sequence was computed for the point. Using the notation of Mahowald-Rezk we let

$$R^{s,t} = \mathbb{Z}_2[a_1, a_3, \Delta^{-1}][[\pi_1, \pi_2, \dots]][\zeta]/(2\zeta)$$

be the bigraded ring such that  $\zeta$  has bidegree  $(1, 0)$ . Assigning odd weight to  $a_1, a_3, \zeta$ , we can identify  $E_2 = H^*(\mathbb{Z}/2\mathbb{Z}, E^*BSpin)$  with the even weight part of  $R^{s,t}$ . Thus we have

$$E_2 \cong \mathbb{Z}_2[a_1^2, a_1a_3, a_3^2, \Delta^{-1}][[\pi_1, \pi_2, \dots]][x]/(2x),$$

where  $x = \zeta a_3^3$ . Since the Pontryagin classes are permanent cycles the spectral sequence converges to

$$TMF_0(3)^*[[\pi_1, \pi_2, \dots]]$$

and the theorem 1.1 is proven.  $\square$

It shall be noticed that all properties of the Pontryagin classes are inherited from the complex classes, for instance we have

**Corollary 2.4.** *The Pontryagin classes are stable, natural and satisfy the Cartan formula*

$$\pi_t(\xi \oplus \eta) = \pi_t(\xi)\pi_t(\eta) \in TMF_0(3)^*X[t]$$

### 3. CUBICAL STRUCTURES AND THE DEFECT CLASS

After the construction of the Pontryagin classes we now consider the remaining generator  $r$  of the  $TMF_1(3)$ -cohomology of  $BString$ . In [Lau] this class has been specified by the property that it maps to a generator under the canonical map to the cohomology of  $K(\mathbb{Z}, 3)$ . However, not all of these choices will have a lift to  $TMF_0(3)$ . In this section we will give a specific choice of the class whose appearance as a defect class allows the desired lift.

Let  $C$  be an elliptic curve. Recall from [AHS01] that a cubical structure on the line bundle  $\mathcal{I}(0)$  is a section of a line bundle  $\Theta^3(\mathcal{I}(0))$  over  $C^3$  satisfying certain properties. The theorem of the cube implies that each elliptic curve has a unique such cubical structure. On the other hand, if  $\hat{C}$  denotes the formal group of the elliptic curve, and  $E$  is an elliptic cohomology theory such that  $\hat{C} \cong E^0\mathbb{C}P^\infty$ , then we have

**Theorem 3.1.** [AHS01] *Cubical structures on the restriction of  $\mathcal{I}(0)$  to  $\hat{C}^3$  are in bijection with ring spectrum maps  $MU\langle 6 \rangle \rightarrow E$ .*

In particular, the cubical structure on  $C$  defines a distinguished cubical structure on  $\hat{C}$ , and a distinguished ring spectrum map  $\sigma : MU\langle 6 \rangle \rightarrow E$ .

We want to use the Thom isomorphism to obtain a class in  $E^*BU\langle 6 \rangle$ , therefore we need the choice of a Thom class. Each complex orientation of  $E$  induces a Thom class, and defining a complex orientation of  $E$  is equivalent to defining a coordinate on the formal group  $\hat{C}$ . This coordinate defines a trivialization of the bundle  $\mathcal{I}(0)$ . Using this trivialization, cubical structures on the restriction of  $\mathcal{I}(0)$  to  $\hat{C}^3$  correspond to cubical structures on the trivial line bundle over  $\hat{C}^3$  (or equivalently power series in three variables satisfying certain properties). Ando, Hopkins and Strickland show that the latter can be identified with ring spectrum maps  $BU\langle 6 \rangle_+ \rightarrow E$ . This uses composition with the map

$$\prod (1 - L_i) : (\mathbb{C}P^\infty)^3 \rightarrow BU\langle 6 \rangle$$

and the isomorphism  $E^*(\mathbb{C}P^\infty)^3 \cong E^*[[x_0, x_1, x_2]]$  after a choice of coordinate.

If  $C \rightarrow S$  is given in Weierstrass form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

that is, as a subset of  $\mathbb{P}^2$ , the cubical structure on  $\mathcal{I}(0)$  is given by the section  $s(c_0, c_1, c_2) = t(c_0, c_1, c_2)d(X/Y)_0$  of  $\Theta^3(\mathcal{I}(0))$  over  $C^3$ , where

$$t(c_0, c_1, c_2) = \frac{\begin{vmatrix} X_0 & Z_0 \\ X_1 & Z_1 \end{vmatrix} \begin{vmatrix} X_1 & Z_1 \\ X_2 & Z_2 \end{vmatrix} \begin{vmatrix} X_2 & Z_2 \\ X_0 & Z_0 \end{vmatrix}}{\begin{vmatrix} X_0 & Y_0 & Z_0 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} Z_0 Z_1 Z_2}.$$

Here  $c_i = [X_i : Y_i : Z_i]$ , and  $d(X/Y)_0$  is a section of the bundle  $p^*\omega$ , where  $P : C^3 \rightarrow S$ , and  $t$  is a function on  $C^3$  with divisor

$$D = -\sum [c_i = 0] + \sum [c_i + c_j = 0] - [c_0 + c_1 + c_2 = 0].$$

The function  $t$  is a trivialization of the corresponding line bundle  $\mathcal{I}_D$ , and one has an isomorphism

$$\Theta^3(\mathcal{I}(0)) \cong \mathcal{I}_D \otimes p^*\omega.$$

Now we consider the formal group of the Weierstrass curve: the zero section of  $C$  is  $[X : Y : Z] = [0 : 1 : 0]$ , and there is a natural choice for the coordinate on  $\hat{C}$ : it is the function  $x = X/Y$ . We also denote  $z = Z/Y$ , equivalently we use the affine chart  $Y = 1$ . On the formal group one can write  $z$  as a formal power series

$$z(x) = x^3 - a_1 x^4 + (a_1^2 + a_2) x^5 - (a_1^3 + 2a_1 a_2 + a_3) x^6 + \dots$$

in the coordinate, and one also has a formal expansion of the addition on  $\hat{C}$ :

$$x_0 +_F x_1 = x_0 + x_1 + a_1 x_0 x_1 - a_2 (x_0^2 x_1 + x_0 x_1^2) + \dots$$

The trivialization  $x$  of the restriction of  $\mathcal{I}(0)$  leads to corresponding trivializations  $d(X/Y)_0$  of the restriction of  $p^* \omega$  and

$$u(x_0, x_1, x_2) = \frac{(x_0 +_F x_1)(x_1 +_F x_2)(x_2 +_F x_0)}{x_0 x_1 x_2 (x_0 +_F x_1 +_F x_2)}$$

of the restriction of  $\mathcal{I}_D$ . Also

$$t = \frac{\begin{vmatrix} x_0 & z_0 \\ x_1 & z_1 \end{vmatrix} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \begin{vmatrix} x_2 & z_2 \\ x_0 & z_0 \end{vmatrix}}{\begin{vmatrix} x_0 & 1 & z_0 \\ x_1 & 1 & z_1 \\ x_2 & 1 & z_2 \end{vmatrix} z_0 z_1 z_2}$$

becomes a quotient of formal power series in  $x_0, x_1, x_2$ , since  $z_i = z(x_i)$ .

The coordinate  $x$  on the formal group can be considered as a ring spectrum map  $x : MU \rightarrow E$ , and we also denote by  $x$  the composition with the obvious map  $MU\langle 6 \rangle \rightarrow MU$ . It follows that the cubical structure on the trivial line bundle corresponds to the ring spectrum map

$$\frac{\sigma}{x} : BU\langle 6 \rangle_+ \rightarrow E.$$

This class will be denoted by  $r_U$  in the sequel.

**Proposition 3.2.** *The cubical structure which corresponds to  $r_U = \frac{t(x_0, x_1, x_2)}{u(x_0, x_1, x_2)}$  is given by*

$$1 - (a_1 a_2 - 3a_3) x_0 x_1 x_2 - (a_1 a_3 - a_2^2 + 5a_4) (x_0^2 x_1 x_2 + x_0 x_1^2 x_2 + x_0 x_1 x_2^2) + \dots$$

This can be calculated with a computer algebra system. However, later we will only be interested in the coefficient in front of  $x_0 x_1 x_2$  which can quickly be calculated by hand.

The spectrum  $E = TMF_1(3)$  is an elliptic spectrum and the corresponding elliptic curve has a canonical Weierstrass form. Therefore we obtain a distinguished class  $r_U = \frac{\sigma}{x} \in E^* BU\langle 6 \rangle$ .

There is a commutative diagram of fibrations:

$$\begin{array}{ccccc} K(\mathbb{Z}, 3) & \xrightarrow{i} & BString & \longrightarrow & BSpin \\ \downarrow \cdot 2 & & \downarrow c & & \downarrow \\ K(\mathbb{Z}, 3) & \xrightarrow{j} & BU\langle 6 \rangle & \longrightarrow & BSU \end{array}$$

where  $c$  is induced from complexification of vector bundles. We denote

$$r = c^* r_U \in E^* BString.$$

**Remark 3.3.** It is interesting to note that  $MU\langle 6 \rangle \rightarrow E$  factors through the realification map  $MU\langle 6 \rangle \rightarrow MString$ , so that both complexification and realification appear in the definition of the class  $r$ . However, the class  $r$  does not factor through the map  $BString \rightarrow BString$  which is given by the composition of complexification and realification — this would be multiplication by 2 in the H-space, and so cannot produce a generator in cohomology.

Next we will show that the class  $r$  defines a generator in the cohomology of  $BString$ . Let  $\hat{E} = L_{K(2)}E$ , and let  $r$  be the image of  $r$  under the map  $E^*BString \rightarrow \hat{E}^*BString$  induced by the localization map  $E \rightarrow \hat{E}$ .

**Theorem 3.4.** *Let  $p_i$  be the Pontryagin classes. Then the map*

$$E^*[[r, \pi_1, \pi_2, \dots]] \longrightarrow E^*BString$$

*is injective. Moreover it is an isomorphism after completion at the invariant prime ideal  $I_2 = (2, a_1)$*

$$E^*(BString)_{I_2}^\wedge \cong \hat{E}^*[[r, \pi_1, \pi_2, \dots]] \cong \hat{E}^*(BString).$$

*Proof.* We denote the image of  $r$  under the natural map  $\hat{E} \rightarrow K(2)$  by the same name  $r \in K(2)^*BString$ . By [KLW04] there is an epimorphism of Hopf algebras

$$p : K(2)^*BString \rightarrow K(2)^*K(\mathbb{Z}, 3)$$

which arises from the diagram

$$K(2)^*BString \xrightarrow{i^*} K(2)^*K(\mathbb{Z}, 3) \xleftarrow{(\cdot 2)^*} K(2)^*K(\mathbb{Z}, 3)$$

as both maps have the same image and the right map is a monomorphism. Since  $r = c^*r_U$ , we have

$$p(r) = j^*r_U.$$

Recall from [RW80] that the Hopf algebra  $K(2)_*K(\mathbb{Z}, 3)$  is a divided power algebra, and the dual Hopf algebra  $K(2)^*K(\mathbb{Z}, 3)$  is a power series algebra on one generator. By [Lau] it is enough to show that  $p(r)$  is a free topological generator for  $K(2)^*K(\mathbb{Z}, 3)$ , that is,

$$K(2)^*K(\mathbb{Z}, 3) \cong K(2)^*[[p(r)]].$$

We explain a result from [Su07] that describes certain topological generators for  $K(2)^*K(\mathbb{Z}, 3)$ : There is a commutative diagram

$$\begin{array}{ccccc} & & (\mathbb{CP}^\infty)^3 & & \\ & & \downarrow & & \\ K(\mathbb{Z}, 3) & \longrightarrow & \underline{BP}\langle 1 \rangle_6 & \longrightarrow & \underline{BP}\langle 1 \rangle_4 \\ \downarrow = & & \downarrow & & \downarrow \\ K(\mathbb{Z}, 3) & \xrightarrow{j} & \underline{ku}_6 = BU\langle 6 \rangle & \longrightarrow & \underline{ku}_4 = BSU \end{array}$$

Here we have localized at the prime 2, and we denote the spaces in an Omega-Spectrum  $E$  by  $\underline{E}_n$ , so that in a ring spectrum we have maps

$$\mu : \underline{E}_m \times \underline{E}_n \rightarrow \underline{E}_{m+n}$$



There is the standard inclusion  $\mathbb{CP}^\infty = \underline{MU}(1)_2 \rightarrow \underline{MU}_2$  and standard ring spectra maps  $MU \rightarrow BP \rightarrow BP\langle 1 \rangle$  through which the 2-typicalization of the complex orientation  $1 - L$  of  $ku$  factorizes. Hence the 2-typicalization of the map

$$\prod (1 - L_i) : (\mathbb{CP}^\infty)^3 \rightarrow BU\langle 6 \rangle$$

which has been used above for the identification with the cubical structures appears in the diagram as the composition of the two vertical maps in the middle column. The map and its typicalization coincide on the product  $(\mathbb{CP}^1)^3$  since they are products of Euler classes. Hence, the images of the canonical generator  $\beta_1^{\otimes 3} \in K(2)_*(\mathbb{CP}^\infty)^3$  agree.

Su proves [Su07], Proposition 4.2, that a class in  $K(2)^* \underline{BP}\langle 1 \rangle_6$  maps to a generator of  $K(\mathbb{Z}, 3)$  if its restriction to  $(\mathbb{CP}^\infty)^3$  pairs with  $\beta_1^{\otimes 3} \in K(2)_*(\mathbb{CP}^\infty)^3$ .

Now we consider  $r_U$ . We claim that its image under  $j^*$  is a generator. This follows from the fact that its image in  $K(2)^*(\mathbb{CP}^\infty)^3$  has a suitable coefficient in front of  $x_0 x_1 x_2$  by 3.2 and  $x_0 x_1 x_2$  is dual to  $\beta_1^{\otimes 3} \in K(2)_*(\mathbb{CP}^\infty)^3$ . This shows the claim about the  $\hat{E}$  cohomology of  $BString$ . The  $E$  cohomology follows from [Lau]4.5.  $\square$

We next like to show that the class  $r$  comes from a class in  $TMF_0(3)^* BString$ . For that, we consider the first stages of the equivariant Whitehead tower for  $B\mathbb{U}$ .

Let  $B\mathbb{U}\langle 4 \rangle = BS\mathbb{U}$  be the homotopy fibre of the first Real Chern class

$$c_1 : B\mathbb{U} \rightarrow K(\mathbb{Z}, \mathbb{C}),$$

let  $B\mathbb{U}\langle 6 \rangle$  be the fibre of

$$c_2 : B\mathbb{U}\langle 4 \rangle \rightarrow K(\mathbb{Z}, \mathbb{C}^2),$$

and let  $B\mathbb{U}\langle 8 \rangle$  be the fibre of

$$c_3 : B\mathbb{U}\langle 6 \rangle \rightarrow K(\mathbb{Z}, \mathbb{C}^3).$$

Note that non-equivariantly  $B\mathbb{U}\langle 6 \rangle$  coincides with  $BU\langle 6 \rangle$  and the first five homotopy groups vanish. Taking  $\mathbb{Z}/2$ -fixed points, the spaces  $B\mathbb{U}\langle 2k \rangle^{\mathbb{Z}/2}$  are  $(k-1)$ -connected. We have  $B\mathbb{U}^{\mathbb{Z}/2} = BO$ , and fibrations (c.[Dug05])

$$\begin{aligned} BSO = BS\mathbb{U}^{\mathbb{Z}/2} &\rightarrow BO = B\mathbb{U}^{\mathbb{Z}/2} \rightarrow K(\mathbb{Z}, \mathbb{C})^{\mathbb{Z}/2} = K(\mathbb{Z}/2, 1) \\ B\mathbb{U}\langle 6 \rangle^{\mathbb{Z}/2} &\rightarrow BSO = BS\mathbb{U}^{\mathbb{Z}/2} \rightarrow K(\mathbb{Z}, \mathbb{C}^2)^{\mathbb{Z}/2} = K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4) \\ B\mathbb{U}\langle 8 \rangle^{\mathbb{Z}/2} &\rightarrow B\mathbb{U}\langle 6 \rangle^{\mathbb{Z}/2} \rightarrow K(\mathbb{Z}, \mathbb{C}^3)^{\mathbb{Z}/2} = K(\mathbb{Z}/2, 3) \times K(\mathbb{Z}/2, 5). \end{aligned}$$

Since  $B\mathbb{U}\langle 6 \rangle^{\mathbb{Z}/2}$  is 2-connected, the map  $BSO \rightarrow K(\mathbb{Z}/2, 2)$  in the second line is  $w_2$ , and since in the third line  $B\mathbb{U}\langle 6 \rangle^{\mathbb{Z}/2} \rightarrow K(\mathbb{Z}/2, 3)$  is an isomorphism on  $\pi_3$ , the map  $BSO \rightarrow K(\mathbb{Z}, 4)$  in the second line is  $p_1$ . Therefore  $B\mathbb{U}\langle 6 \rangle^{\mathbb{Z}/2}$  is the homotopy fiber of  $p_1 : BSpin \rightarrow K(\mathbb{Z}, 4)$ . It follows that the map  $c : BString \rightarrow BU\langle 6 \rangle$  factors through a natural map  $BString \rightarrow B\mathbb{U}\langle 6 \rangle^{\mathbb{Z}/2}$ , or equivalently, if we equip  $BString$  with the trivial  $\mathbb{Z}/2$ -action, we have an equivariant map

$$BString \rightarrow B\mathbb{U}\langle 6 \rangle.$$

Also the map

$$\prod (1 - L_i) : (\mathbb{CP}^\infty)^3 \rightarrow BU\langle 6 \rangle$$

is equivariant. From [AHS01] it follows that the ordinary homology of  $BU\langle 6 \rangle$  is generated as an algebra by its image. Hence  $BU\langle 6 \rangle$  has the projective property in the sense of [KW13].

**Lemma 3.5.** *The forgetful map*

$$B\mathbb{P}\mathbb{R}^{k(1+\alpha)}BU\langle 6 \rangle \xrightarrow{\cong} BP^{2k}BU\langle 6 \rangle$$

*is an isomorphism.*

*Proof.* The Real spectrum  $B\mathbb{P}\mathbb{R}$  satisfies the strong completion theorem, that is, we can replace it with  $\hat{B}\mathbb{P}\mathbb{R} = F(E\mathbb{Z}/2_+, B\mathbb{P}\mathbb{R})$  by [HK01]4.1(1). Now the argument follows theorem 2.3 in [KW13] and we repeat it here: let  $Z$  be  $\Omega\Sigma(\mathbb{C}P^\infty)^3$  and set  $Y = BU\langle 6 \rangle$ . The  $\mathbb{Z}/2$  space  $Z$  admits an equivariant James filtration and hence splits into a wedge of suspensions of  $\mathbb{C}P^\infty$ . Using the Real cellular decomposition of  $\mathbb{C}P^\infty$  we see that  $B\mathbb{P}\mathbb{R} \wedge Z$  is a free  $B\mathbb{P}\mathbb{R}$ -module of finite type:

$$B\mathbb{P}\mathbb{R} \wedge Z \cong \bigvee \Sigma^{k_i(1+\alpha)} B\mathbb{P}\mathbb{R}.$$

Choose a subsequence  $\beta_1, \beta_2, \dots$  of  $k_1, k_2, \dots$  and obtain an equivariant map

$$\bigvee \Sigma^{\beta_i(1+\alpha)} B\mathbb{P}\mathbb{R} \longrightarrow Y \wedge B\mathbb{P}\mathbb{R}$$

which is a non equivariant homotopy equivalence. This gives an equivariant equivalence of  $B\mathbb{P}\mathbb{R}$ -module spectra

$$\bigvee \Sigma^{\beta_i(1+\alpha)} E\mathbb{Z}/2_+ \wedge B\mathbb{P}\mathbb{R} \longrightarrow E\mathbb{Z}/2_+ \wedge Y \wedge B\mathbb{P}\mathbb{R}$$

and hence

$$B\mathbb{P}\mathbb{R}^{*,*}Y \cong \hat{B}\mathbb{P}\mathbb{R}^{*,*}Y \cong B\mathbb{P}\mathbb{R}^{*,*}(Y \wedge E\mathbb{Z}/2_+) \cong B\mathbb{P}\mathbb{R}^{*,*}\langle\langle\gamma_1, \gamma_2, \dots\rangle\rangle$$

with generators  $\gamma_i$  of degree  $\beta_i(1+\alpha)$  which also freely generate the non equivariant cohomology.  $\square$

We consider the function  $\mathbb{Z}/2$ -spectra  $\mathbb{F}(?, \mathbb{E})$  whose fixed points  $\mathbb{F}(?, \mathbb{E})^{\mathbb{Z}/2}$  are the spectra of equivariant functions. For the trivial  $\mathbb{Z}/2$ -space  $BString$ , the latter is equivalent to the function spectrum  $F(BString, \mathbb{E}^{\mathbb{Z}/2})$  over the trivial universe. Since we have a strong completion theorem for  $\mathbb{E}$  the homotopy fixed point spectral sequence converges to the homotopy of the fixed point spectrum:

$$E_2 = H^*(\mathbb{Z}/2, E^*(BString)) \Rightarrow \pi_*\mathbb{F}(BString, \mathbb{E})^{\mathbb{Z}/2} \cong (\mathbb{E}^{\mathbb{Z}/2})^*(BString)$$

The class  $r_U : BU\langle 6 \rangle \rightarrow E$  can be lifted to a map  $\tilde{r}_U : BU\langle 6 \rangle \rightarrow BP$  and hence is equivariant up to homotopy by the lemma. When mapping back to  $\mathbb{E}$  we obtain an equivariant representative of  $r_U$  itself. Taking fixed points and composing with the map from  $BString$  we have an equivariant map  $r : BString \rightarrow \mathbb{E}^{\mathbb{Z}/2}$ . It follows that  $r \in E_2^{0,*}$  is a permanent cycle.

*Proof of 1.2.* The multiplicativity of  $r$  under direct sums follows from the multiplicativity of the Thom classes  $\sigma$  and  $x$  respectively.

For the calculation of the  $\widehat{TMF}_0(3)$  cohomology of  $BString$ , we can employ the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(\mathbb{Z}/2\mathbb{Z}, \hat{E}^t BString) \Rightarrow (\hat{E}^{h\mathbb{Z}/2\mathbb{Z}})^* BString.$$

We have

$$\hat{E}^* BString \cong \mathbb{Z}_2[a_1, a_3, \Delta^{-1}][[r, \pi_1, \pi_2, \dots]]_{I_2}^\wedge,$$

and  $\mathbb{Z}/2\mathbb{Z}$  acts by  $a_i \mapsto -a_i, r \mapsto r, \pi_i \mapsto \pi_r$ . Using the same notations as in the *Spin* case we can identify  $E_2 = H^*(\mathbb{Z}/2\mathbb{Z}, \hat{E}^* BString)$  with the even weight part of

$$R^{s,t} = \mathbb{Z}_2[a_1, a_3, \Delta^{-1}][[r, \pi_1, \pi_2, \dots]][\zeta]/(2\zeta)_{I_2}^\wedge$$

and we obtain

$$E_2 \cong \mathbb{Z}_2[a_1^2, a_1 a_3, a_3^2, \Delta^{-1}][[r, \pi_1, \pi_2, \dots]][x]/(2x)_{I_2}^\wedge.$$

We know that  $r$  and the Pontryagin classes are permanent cycles. Hence the spectral sequence converges to

$$\hat{E}^{h\mathbb{Z}/2*}[[r, \pi_1, \pi_2, \dots]]$$

and the second part of the main theorem 1.2 is proven.

It remains to show the formula for the characteristic class  $r \in TMF_0(3)(BString)$ . The elliptic character at the cusp  $\infty$  is the composition  $TMF_0(3) \rightarrow TMF_1(3) \rightarrow K[\frac{1}{3}, \zeta_3]((q))$ . Since we want to compose with the Chern character  $K \rightarrow HP\mathbb{Q}$  to periodic rational cohomology, we can rationalize everywhere. Thus it suffices to consider the map on the coefficients:

$$(TMF_1(3))_{\mathbb{Q}}^{-2n} \rightarrow \left( K\left[\frac{1}{3}, \zeta_3\right]((q)) \right)_{\mathbb{Q}}^{-2n} \cong \mathbb{C}((q))$$

is the  $q$ -expansion of modular forms of degree  $2n$ . We need to compose this with the composition along the left and bottom in the diagram

$$\begin{array}{ccccc} BString & \longrightarrow & BU\langle 6 \rangle & \longrightarrow & BU \\ \downarrow r & & \downarrow r_U = \frac{\sigma}{x} & & \downarrow \psi \\ E^{\mathbb{Z}/2} & \longrightarrow & E & \longrightarrow & HE_{\mathbb{Q}}^*. \end{array}$$

The formula for these orientations in terms of a single complex line bundle can be found in [HBJ92] Appendix I, 5.3 and 6.4: We denote by  $z$  the complex variable. For bundles with vanishing first Pontryagin class, we may replace the function  $\sigma(\tau, z)$  by  $\Phi(\tau, z)$ . Moreover the complex orientation  $MU\langle 6 \rangle \rightarrow MU \rightarrow E$  corresponds to the elliptic genus of level 3, that is, to the function

$$x(z) = \frac{\Phi(\tau, z)\Phi(\tau, -\omega)}{\Phi(\tau, z - \omega)}.$$

Hence we have for the quotient  $r = \frac{\sigma}{x}$  the formula in the theorem.  $\square$

#### APPENDIX A. THE INVARIANCE OF THE PONTRYAGIN CLASSES AND THE DEFECT CLASS

We like to give a geometric proof for the invariance of the Pontryagin classes and the defect class.

Two coordinates  $x, x'$  on the same elliptic curve induce an isomorphism of the corresponding Weierstrass curves with coordinates  $[X : Y : Z]$  and  $[X' : Y' : Z']$  respectively. Such an isomorphism is in general given by

$$X' = u^2 X + rZ, \quad Y' = su^2 X + u^3 Y + tZ, \quad Z' = Z.$$

This implies

$$x' = \frac{u^{-1}x + ru^{-3}z}{1 + tu^{-3}z + su^{-1}x} = g(x),$$

which is the usual description of a coordinate change on a formal group using a power series  $g(x)$  with vanishing constant and invertible linear coefficient.

In particular we are interested in  $E = TMF_1(3)$  and the universal triple  $(C, \omega, P)$  of an elliptic curve with invariant differential and level structure consisting of a point  $P$  of order 3. This is

$$C : Y^2Z + a_1XYZ + a_3YZ^2 = X^3$$

over  $TMF_1(3)_* = \mathbb{Z}_2[a_1, a_3, \Delta^{-1}]$  with  $\omega = \frac{dX}{2Y+a_1X+a_3}$  and  $P = (0, 0)$ .

We also consider

$$C^- : Y^2Z - a_1XYZ - a_3YZ^2 = X^3$$

over  $TMF_1(3)_*$  because  $(C^-, \omega, P = (0, 0))$  is isomorphic to  $(C, \omega, -P)$  via the isomorphism

$$X' = X, Y' = a_1X + Y + a_3Z, Z' = Z.$$

As  $(C, \omega, P)$  is universal, we obtain a corresponding ring automorphism of  $TMF_1(3)_* = \mathbb{Z}_2[a_1, a_3, \Delta^{-1}]$  of order 2: it sends  $a_1 \mapsto -a_1, a_3 \mapsto -a_3$ .

Note that the new coordinates  $[X' : Y' : Z']$  are the coordinates of the negative of the point  $[X : Y : Z]!$

The new complex coordinate is

$$x' = g(x) = \frac{x}{1 + a_1x + a_3z(x)}.$$

But since we have just taken the coordinate of the negative point on the elliptic curve, we also have

$$\bar{x} = [-1](x) = g(x)$$

Let

$$Q(x) = \frac{x}{g(x)} = 1 + a_1x + a_3z(x) \in E^*[[x]].$$

If we consider both coordinates as ring maps  $MU \rightarrow E$  then they are both generators for the free rank 1  $E^*BU$ -module  $E^*MU$ , so that their quotient  $\frac{x'}{x}$  defines an invertible element of  $E^*BU$ . Using the complex coordinate  $x$ , we have isomorphisms and an injection

$$E^*(BU) \cong E^*[[c_1, c_2, \dots]] \rightarrow E^*(BU(1)^\infty) \cong E^*[[x_1, x_2, \dots]],$$

where each  $c_k$  is mapped to the  $k$ -th elementary symmetric polynomial in the  $x_i$ . This is induced by the map

$$\sum (L_i - 1) : BU(1)^\infty \rightarrow BU.$$

In  $E^*[[x_1, x_2, \dots]]$ , the element  $\frac{x}{x'}$  corresponds to  $\prod_{k=1}^\infty Q(x_k)$ , which is symmetric in the  $x_k$ , so that it defines an element in  $E^*[[\sigma_j(x_i)]] \cong E^*BU$ .

By composition with  $MU\langle 6 \rangle \rightarrow MU$ , we obtain  $x, x' : MU\langle 6 \rangle \rightarrow E$ , so that together with  $\sigma : MU\langle 6 \rangle \rightarrow E$ , we obtain  $r_U = \frac{\sigma}{x}, r'_U = \frac{\sigma}{x'} : BU\langle 6 \rangle_+ \rightarrow E$ . Let  $\tau$  denote the involutions on  $MU\langle 6 \rangle, BU\langle 6 \rangle_+, E$  respectively. The following diagrams commute up to homotopy:

$$\begin{array}{ccc} MU & \xrightarrow{x} & E \\ \downarrow \tau & \searrow x' & \downarrow \tau \\ MU & \xrightarrow{x} & E \end{array} \quad \begin{array}{ccc} MU\langle 6 \rangle & \xrightarrow{\sigma} & E \\ \downarrow \tau & \searrow \sigma & \downarrow \tau \\ MU\langle 6 \rangle & \xrightarrow{\sigma} & E, \end{array}$$

therefore  $r_U : BU\langle 6 \rangle_+ \rightarrow E$  is up to homotopy equivariant — both compositions  $\tau \circ r_U, r_U \circ \tau$  are equal to  $r'_U$ .

We can also consider the restriction to  $BString$  under the map  $BString \rightarrow BSO \xrightarrow{\mathcal{C}} BU$ . Unstably, we have

$$E^*BU(2N+1) \cong E^*[[c_1, c_2, \dots, c_{2N+1}]]$$

and

$$E^*BU(2N+1) \cong E^*B\mathbb{T}'^W$$

where  $\mathbb{T}'$  is a  $2N+1$ -dimensional maximal torus in  $U(2N+1)$  and  $W \cong \Sigma_{2N+1}$  is the Weyl group for  $U(2N+1)$ , acting on  $E^*B\mathbb{T}' \cong E^*[[y_1, \dots, y_{2N+1}]]$  by permuting the  $y_i$ . The Chern classes  $c_j = \sigma_j(y_i)$  are the elementary symmetric polynomials of the  $y_i$ .

Comparing the unitary and special orthogonal groups and their maximal tori, we obtain the diagram

$$\begin{array}{ccc} E^*BU(2N+1) \cong E^*[[c_j]] & \longrightarrow & E^*BSO(2N+1) \\ \downarrow & & \downarrow \\ E^*B\mathbb{T}' \cong E^*[[y_j]] & \longrightarrow & E^*B\mathbb{T} \cong E^*[[x_j]] \end{array}$$

The maximal torus in  $SO(2N+1)$  consists of matrices

$$A = \begin{pmatrix} R_{\phi_1} & & & \\ & R_{\phi_2} & & \\ & & \dots & \\ & & & 1 \end{pmatrix},$$

where  $R_\phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$ . The standard maximal torus in  $U(N)$  consists of diagonal matrices. Under conjugation by

$$\begin{pmatrix} T & & & \\ & T & & \\ & & \dots & \\ & & & 1 \end{pmatrix},$$

where  $T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ , the matrix  $A$  corresponds to the diagonal matrix  $\text{diag}(e^{i\phi_1}, e^{-i\phi_1}, e^{i\phi_2}, e^{-i\phi_2}, \dots, 1)$ , so that we can assume that the map  $B\mathbb{T} \rightarrow B\mathbb{T}'$  is induced by

$$U(1)^N \rightarrow U(1)^{2N+1}, \quad (z_1, \dots, z_N) \mapsto (z_1, z_1^{-1}, \dots, z_N, z_N^{-1}, 1).$$

It follows that in cohomology,  $y_{2k-1} \mapsto x_k, y_{2k} \mapsto \overline{x_k}, y_{2N+1} \mapsto 0$ .

For the image of the class  $\frac{x}{x'}$ , we have that  $Q(x_{2k-1}) \mapsto Q(y_k) = \frac{y_k}{g(y_k)}$  and  $Q(x_{2k}) \mapsto Q(\overline{y_k}) = \frac{\overline{y_k}}{g(\overline{y_k})} = \frac{g(y_k)}{y_k}$ . It follows that  $\prod_{k=1}^\infty Q(x_k) \mapsto 1$ .

Conclusion: the image of  $\frac{x}{x'}$  in  $E^*BString$  is trivial, so that  $\mathbb{Z}/2\mathbb{Z}$  acts trivially on  $r \in E^*(BString)$ .

Since the action of  $\mathbb{Z}/2\mathbb{Z}$  on  $E^*B\mathbb{T}$  sends each  $x_i \mapsto \overline{x_i}$ , we see that all  $x_i \cdot \overline{x_i}$  and therefore also all Pontryagin classes are invariant elements of  $E^*(BString)$ .

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